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Some new bounds on T_r-choosability

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Abstract

List T-colouring is a generalisation of list colouring in which the differences between adjacent colours must not lie in the set T. We present a conjecture giving an upper bound on the T_r-choosability $T_r\text{-ch}(G)$ (where $T_r = \{0, 1, \ldots, r\}$) in terms of r and $\text{ch}(G)$ which, if true, is tight for all values of r and $\text{ch}(G)$, and we prove the bound in the case $\text{ch}(G) = 2$. We also prove the conjecture with the colouring number $\text{col}(G)$ in place of $\text{ch}(G)$, and use this result in conjunction with a theorem of Alon to establish an exponential upper bound on $T_r\text{-ch}(G)$ in terms of r and $\text{ch}(G)$.

Keywords: T-Colouring; List colouring; Frequency assignment

1. Introduction and results

List T-colourings of graphs are a combination of two generalisations of ordinary graph (vertex-)colouring. The first of these is *list colouring*, formulated by Vizing [11] and independently by Erdős et al. [3]. Given a graph $G = (V, E)$, a *list assignment* is a function L which assigns to each vertex $v \in V$ a list (set) $L(v) \subseteq \mathbb{Z}$ of colours. An *L-colouring* of G is then a function $c : V \to \mathbb{Z}$ such that $c(v) \in L(v)$ for all $v \in V$, and $c(v) \neq c(w)$ for all $vw \in E$. If an L-colouring of G exists for every list assignment L such that $|L(v)| = k$ for all $v \in V$, then G is said to be *k-choosable*, and the *choosability* $\text{ch}(G)$ of G is the smallest k such that G is k-choosable.

The second generalisation is graph *T-colouring*, whose study was initiated by Hale [5] as a model for the frequency assignment problem. Given a graph $G = (V, E)$ and a set T of non-negative integers, a *T-colouring* of G is a function $c : V \to \mathbb{Z}$ such that $|c(v) - c(w)| \notin T$ for all $vw \in E$. Usually, T is assumed to contain 0; and in particular, $T_r = \{0, 1, \ldots, r\}$, so that a T_r-colouring of G has $|c(v) - c(w)| \geqslant r + 1$ for all $vw \in E$.

Combining these concepts leads naturally to *list T-colourings*, introduced by Tesman [9]. Given G, T and L as above, an *L-T-colouring* of G is a function $c : V \to \mathbb{Z}$ such that both $c(v) \in L(v)$ for all $v \in V$, and $|c(v) - c(w)| \notin T$ for all $vw \in E$. If an L-T-colouring exists for every list assignment L such that $|L(v)| = k$ for all $v \in V$, then G is said to be *T-k-choosable*, and the *T-choosability* $T\text{-ch}(G)$ of G is the smallest k for which G is T-k-choosable.

List T-colourings of graphs were studied extensively by Tesman [9,10], and more recently by Alon and Zaks [2]. The exact value of the T_r-choosability $T_r\text{-ch}(G)$ is computed in [9] for G a complete graph, a tree, or an odd cycle. For G an even cycle, a conjecture of Alon and Zaks [2] on the value of $T_r\text{-ch}(G)$ was proved by Sitters [8].

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A common generalisation of list T_r-colourings and the well-known *channel assignment problem* (for example, see [7]) was studied by Král' and Škrekovski in [6], under the name *list channel assignment problem*. An instance consists of a triple (G, L, w), with G and L as above, and a function $w : E \to \mathbb{N}$ assigning weights to the edges of G. A colouring must satisfy $|c(u) - c(v)| \geqslant w(uv)$ for all $uv \in E$; thus list T_r-colouring corresponds to the list channel assignment problem with $w(e) = r + 1$ for all $e \in E$. In a later paper, Fiala et al. [4] extend this further and define the *generalised list T-colouring*. This problem involves a triple (G, L, t) where each edge $e \in E$ has its own set $t(e)$ of 'forbidden differences' for the edge e, and reduces to list T-colouring when $t(e) = T$ for all $e \in E$. The main result in both [6,4] is a Brooks-type theorem for the respective colouring problem.

In this paper, we consider the problem of finding an upper bound for T_r-ch(G) given only the values of r and ch(G). A lower bound of this form is presented in [2]:

Theorem 1.1 (*Alon and Zaks [2]*). *For all $r \in \mathbb{N}$ and all graphs G,*

$$T_r\text{-ch}(G) \geqslant (r + 1)(\text{ch}(G) - 1) + 1.$$

Tesman [9] observes that T-ch$(G) \leqslant |T| \cdot \Delta(G) + 1$; taking $T = T_r$ and G to be a complete graph, we see that these lower and upper bounds agree, showing that Theorem 1.1 is tight.

We now present the conjecture which is the focus of this paper, before going on to describe its motivation and some partial results in this direction.

Conjecture 1.2. For all $r \in \mathbb{N}$ and all graphs G,

$$T_r\text{-ch}(G) \leqslant (2r + 1)(\text{ch}(G) - 1) + 1.$$

Clearly, if Conjecture 1.2 holds as well as Theorem 1.1, then given any r and G as above, knowing ch(G) would determine T_r-ch(G) to within a factor of 2.

We will prove two main results related to the above conjecture. The first relates T_r-ch(G) to the degeneracy of the graph G. G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. The *colouring number* col(G) is the smallest k such that G is $(k-1)$-degenerate; that is,

$$\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H).$$

Conjecture 1.2 is motivated by the following theorem:

Theorem 1.3. *For all $r \in \mathbb{N}$ and all graphs G,*

$$T_r\text{-ch}(G) \leqslant (2r + 1)(\text{col}(G) - 1) + 1,$$

and this bound is tight for all values of r and col(G).

Observe that T_0-ch(G)=ch(G), and so setting $r=0$ in Theorem 1.3 yields the well-known result that ch$(G) \leqslant$ col(G). On the other hand, this last inequality shows that Theorem 1.3 is a weakening of Conjecture 1.2.

Theorem 1.3 is proved in Section 2 and is then used in conjunction with a result of Alon [1], which links ch(G) to the minimum degree $\delta(G)$, to prove an exponential bound on T_r-ch(G) in terms of r and ch(G):

Theorem 1.4. *For all $r \in \mathbb{N}$ and all k-choosable graphs G,*

$$T_r\text{-ch}(G) \leqslant (2r + 1)\frac{(k^2 + 1)^2}{(\log_2 \mathrm{e})^2} 4^{k+1} + 1.$$

Though this is a very long way from the bound of Conjecture 1.2, it at least establishes that some bound of the required form does exist.

Our second main result is to prove Conjecture 1.2 in the case $\mathrm{ch}(G)=2$, which we do in Section 3:

Theorem 1.5. *For all $r \in \mathbb{N}$ and all 2-choosable graphs G,*

$$T_r\text{-}\mathrm{ch}(G) \leqslant 2r+2.$$

We restrict our attention throughout to connected graphs, since for a disconnected graph G, the value of each of $\mathrm{ch}(G)$, $\mathrm{col}(G)$ and T_r-$\mathrm{ch}(G)$ is simply the maximum of the values of the corresponding parameter over the components of G.

2. T_r-choosability and degeneracy

Proof of Theorem 1.3. The bound in Theorem 1.3 is established by induction on $|V(G)|$. The result is trivial if $|V(G)|=1$; so assume $|V(G)|\geqslant 2$, write $d=\mathrm{col}(G)-1$, and let the vertices of G be assigned lists of size $(2r+1)d+1$. Since G is d-degenerate, it has a vertex v of degree at most d. By the inductive hypothesis, since $\mathrm{col}(G-v)\leqslant\mathrm{col}(G)$, $G-v$ can be T_r-coloured from its lists; now we seek a colour for v. For each neighbour w of v, the colour we choose for v must not lie in the interval $[c(w)-r, c(w)+r]$, which has $2r+1$ elements. But since the list $L(v)$ has more than $(2r+1)d$ elements, we can always choose some colour for v. This shows that T_r-$\mathrm{ch}(G)\leqslant(2r+1)d+1$.

To show that the bound can be attained, we need a d-degenerate graph whose vertices are assigned lists of size $(2r+1)d$, for which no T_r-colouring exists. The graph K_1 trivially suffices if $d=0$, so assume $d\geqslant 1$. Let $t=(2r+1)d$ and take $G=K_{d,t^d}$, with vertex set $V=\{u_0,u_1,\ldots,u_{d-1}\}\cup\{v_{i_0,i_1,\ldots,i_{d-1}} : i_s\in[0,t-1],\ s=0,\ldots,d-1\}$. Assign the following lists:

$$L(u_s)=[2st,(2s+1)t-1],$$

$$L(v_{i_0,i_1,\ldots,i_{d-1}})=\bigcup_{0\leqslant s\leqslant d-1}[2st+i_s-r, 2st+i_s+r].$$

The intervals in the above union are disjoint, as $(2(s+1)t+i_{s+1}-r)-(2st+i_s+r)\geqslant 2t-(t-1)-2r=(2r+1)d-(2r-1)\geqslant 2>0$, and hence $|L(v)|=t$ for all $v\in V$. But whichever choice of colours $c(u_s)=2st+i_s$ we make for the vertices $u_0,u_1,\ldots,u_{d-1}$, we find that $L(v_{i_0,i_1,\ldots,i_{d-1}})$ contains precisely those colours forbidden for the vertex $v_{i_0,i_1,\ldots,i_{d-1}}$, and so there is no T_r-colouring of G. $\square$

Observe that with $t=(2r+1)d$ and $G=K_{d,t^d}$ as in the above proof, $t^d\geqslant d^d$ and so G is not d-choosable, as noted in [3]. Hence $\mathrm{ch}(G)=\mathrm{col}(G)=d+1$, which implies that if Conjecture 1.2 is true, it is also tight for all values of r and $\mathrm{ch}(G)$.

Theorem 1.4 is now an easy consequence of the above result, in view of the following theorem of Alon [1] which effectively bounds the value of $\mathrm{col}(G)$ in terms of $\mathrm{ch}(G)$:

Theorem 2.1 (*Alon [1]*). *If $s\in\mathbb{N}$ and G is a graph whose minimum degree $\delta(G)$ satisfies*

$$\delta(G)>\frac{4(s^2+1)^2}{(\log_2 \mathrm{e})^2}2^{2s},$$

then $\mathrm{ch}(G)>s$.

Proof of Theorem 1.4. Writing $\mathrm{ch}(G)=k$, and noting that $\mathrm{ch}(H)\leqslant k$ for all subgraphs $H\subseteq G$, we can use Theorem 2.1 to bound $\mathrm{col}(G)$:

$$\mathrm{col}(G)=1+\max_{H\subseteq G}\delta(H)\leqslant\frac{4(k^2+1)^2}{(\log_2 \mathrm{e})^2}2^{2k}.$$

Hence, applying Theorem 1.3, we obtain the result of Theorem 1.4. $\square$

Estimating the bound in Theorem 1.4 as $\mathrm{ch}(G) = k \to \infty$, we obtain

$$T_r\text{-ch}(G) = \mathrm{O}(rk^4 4^k).$$

Even in the case of 2-choosable graphs G, the bound given by Theorem 1.4 is approximately $T_r\text{-ch}(G) \leqslant 1538r + 770$, which is quite some way from the correct bound $T_r\text{-ch}(G) \leqslant 2r + 2$ which we prove in the next section.

3. Proof of Theorem 1.5

The proof of Theorem 1.5 relies on Rubin's characterisation of 2-choosable graphs [3], for which we need the following two definitions.

The *core* of a graph G is obtained by successively removing vertices of degree 1 until none remain. If G is assigned lists of size 2, and we remove a vertex v of degree 1 and colour $G - v$ from its lists, we can always choose a colour for v which differs from that of its neighbour. By iterating this 'pruning' process, we see that G is 2-choosable iff its core is 2-choosable.

We define the *Θ-graph* $\Theta_{a,b,c}$ to consist of two distinguished vertices u, w connected by three paths P_1, P_2 and P_3 of lengths a, b and c, respectively. Two examples are shown in Fig. 1.

Now we can state the theorem characterising 2-choosable graphs:

Theorem 3.1 (*Rubin [3]*). *A connected graph G is 2-choosable iff its core is K_1, or C_{2m+2} or $\Theta_{2,2,2m}$ for some $m \geqslant 1$.*

For the two lemmas to follow we will need some additional terminology. Let $P = uv_1v_2 \cdots v_{l-1}w$ be a path of length l, with a list assignment L such that $|L(v_p)| = 2r + 2$ for $1 \leqslant p \leqslant l - 1$. Construct the set $F = F(P) \subseteq L(u) \times L(w)$ as follows: $(x, y) \in F$ iff there is *no* T_r-colouring of P with $c(u) = x$ and $c(w) = y$. Clearly there is a T_r-colouring of P from its lists iff $F \neq L(u) \times L(w)$.

The set F can be characterised as follows: for $x \in L(u)$ and $y \in L(w)$, let

$$I_1(x) = [x - r, x + r],$$

$$I_2(x) = \bigcap_{c \in L(v_1) \setminus I_1(x)} [c - r, c + r],$$

$$I_3(x) = \bigcap_{c \in L(v_2) \setminus I_2(x)} [c - r, c + r],$$

$$\vdots$$

$$I_l(x) = \bigcap_{c \in L(v_{l-1}) \setminus I_{l-1}(x)} [c - r, c + r].$$

Note that $L(v_p) \setminus I_p(x) \neq \emptyset$ for each $p \in [1, l - 1]$, since $|I_p(x)| \leqslant 2r + 1 < 2r + 2 = |L(v_p)|$. If we want to colour P from its lists, starting with $c(u) = x$, the sets $I_p(x)$ are constructed precisely so that the colour we choose for v_1 must not be in $I_1(x)$, the colour we choose for v_2 must not be in $I_2(x)$, and so on. It follows that

$$(x, y) \in F \iff y \in I_l(x). \tag{3.1}$$

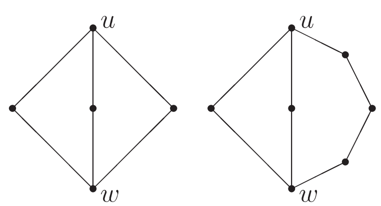

Fig. 1. $\Theta_{2,2,2}$ ($\cong K_{2,3}$), and $\Theta_{2,2,4}$.

Alternatively, we can start at w, and define

$$J_{l-1}(y) = [y-r, y+r],$$

$$J_{l-2}(y) = \bigcap_{c \in L(v_{l-1}) \setminus J_{l-1}(y)} [c-r, c+r],$$

$$\vdots$$

$$J_0(y) = \bigcap_{c \in L(v_1) \setminus J_1(y)} [c-r, c+r].$$

Then

$$(x, y) \in F \iff x \in J_0(y), \tag{3.2}$$

and furthermore, for each $p \in [1, l-1]$,

$$(x, y) \in F \iff L(v_p) \subseteq I_p(x) \cup J_p(y), \tag{3.3}$$

since there is a T_r-colouring of P with $c(u) = x$, $c(v_p) = c$ and $c(w) = y$ if and only if $c \in L(v_p) \setminus [I_p(x) \cup J_p(y)]$.

Since $I_l(x)$ and $J_0(y)$ are intervals, (3.1) and (3.2), respectively, imply the following properties (which we call *convexity*):

$$y_1 < y_2 < y_3 \text{ and } (x, y_1), (x, y_3) \in F \Rightarrow (x, y_2) \in F, \tag{3.4}$$

$$x_1 < x_2 < x_3 \text{ and } (x_1, y), (x_3, y) \in F \Rightarrow (x_2, y) \in F. \tag{3.5}$$

The sets I_p are non-increasing in size with increasing p. To see this, fix $p \in [1, l-1]$ and let a and b be the minimum and maximum elements of $L(v_p) \setminus I_p(x)$. Then

$$\begin{aligned} |I_{p+1}(x)| &= |[a-r, a+r] \cap [b-r, b+r]| \\ &= \max\{(2r+2) - |[a, b]|, 0\} \\ &\leqslant |L(v_p)| - |L(v_p) \setminus I_p(x)|, \end{aligned}$$

which gives

$$|I_{p+1}(x)| \leqslant |L(v_p) \cap I_p(x)| \leqslant |I_p(x)|. \tag{3.6}$$

The following lemma is used later as an auxiliary result, but is stated separately as it may be of interest in itself.

Lemma 3.2. *Let $P = uv_1v_2 \cdots v_{l-1}w$ be a path of length l, whose vertices are assigned lists $L(v)$ such that $|L(v_p)| = 2r+2$ for $1 \leqslant p \leqslant l-1$, and $|L(u)| + |L(w)| > 2r+2$. Then there is a T_r-colouring of P from its lists, such that at least one of u and w is assigned the minimum colour in its list.*

Proof. We use induction on the path length l. Write $L(u) = \{x_1, \ldots, x_i\}$ and $L(w) = \{y_1, \ldots, y_j\}$, with the elements arranged in ascending order in each case. Start by considering the case $l = 1$, i.e. $P = uw$. Since $i + j > 2r + 2$, we have $(x_i - x_1) + (y_j - y_1) > 2r$, and hence $x_i - y_1 > r$ or $y_j - x_1 > r$. In the former case we have a T_r-colouring of P by setting $c(u) = x_i$ and $c(w) = y_1$, and in the latter case, by setting $c(u) = x_1$ and $c(w) = y_j$.

Now assume that $l > 1$, and that the lemma is true for a path of length $l - 1$. Suppose no T_r-colouring of P of the required type exists, so that $(x_s, y_1) \in F$ for each $s \in [1, i]$ and $(x_1, y_t) \in F$ for each $t \in [1, j]$. We will obtain a contradiction.

By (3.1), $\{y_1, \ldots, y_j\} \in I_l(x_1)$ and so, by repeated application of (3.6),

$$|L(v_1) \cap I_1(x_1)| \geqslant |I_2(x_1)| \geqslant |I_l(x_1)| \geqslant j. \tag{3.7}$$

We consider two separate cases.

Case 1: $\max L(v_1) \notin I_1(x_1)$. Then, since $(x_1, y_1) \in F$ and $(x_i, y_1) \in F$,

$$\begin{aligned}
&\max L(v_1) \in J_1(y_1) \quad \text{by (3.3)}\\
\Rightarrow\ &\min L(v_1) \notin J_1(y_1) \quad \text{since } |J_1(y_1)| \leqslant 2r+1 < |L(v_1)|\\
\Rightarrow\ &\min L(v_1) \in I_1(x_i) \quad \text{by (3.3)}\\
\Rightarrow\ &\min L(v_1) \geqslant x_i - r\\
\Rightarrow\ &|L(v_1) \cap I_1(x_1)| \leqslant (2r+1) - (x_i - x_1) \leqslant 2r+2-i.
\end{aligned}$$

Together with (3.7), this shows that $i + j \leqslant 2r+2$, contrary to the hypothesis of the lemma. In this case, the above 'exchange' trick using (3.3) allows us to establish the result without using the inductive hypothesis.

Case 2: $\max L(v_1) \in I_1(x_1)$. This means that $\min L(v_1) \notin I_1(x_1)$ and hence, by (3.3),

$$\min L(v_1) \in J_1(y_t) \quad \text{for } 1 \leqslant t \leqslant j. \tag{3.8}$$

Also, since $\max L(v_1) \leqslant x_1 + r$, we have $|L(v_1) \cap I_1(x_i)| \leqslant (2r+1) - (x_i - x_1) \leqslant 2r+2-i$, and so $L(v_1)$ contains elements $\min L(v_1) = x'_1 < x'_2 < \cdots < x'_i$ such that $x'_s \notin I_1(x_i)$ and hence

$$x'_s \in J_1(y_1) \quad \text{for } 1 \leqslant s \leqslant i. \tag{3.9}$$

Thus, writing $P' = v_1 v_2 \cdots v_{l-1} w$ and $L'(v_1) = \{x'_1, \ldots, x'_i\}$, $L'(w) = L(w) = \{y_1, \ldots, y_j\}$ and $L'(v_p) = L(v_p)$ for $2 \leqslant p \leqslant l-1$, by (3.8) and (3.9) there is no T_r-colouring of P' from its lists such that at least one of v_1 and w is assigned the minimum colour in its list. This contradicts the inductive hypothesis, and so completes the proof of Lemma 3.2. □

The key to the proof of Theorem 1.5 is the following lemma.

Lemma 3.3. *If the vertices of $G = \Theta_{2a,2b,2c}$ are assigned lists $L(v)$ of size $2r+2$ $(a, b, c, r \geqslant 1)$, and $L(u) \neq L(w)$, then there is a T_r-colouring of G from its lists.*

Proof. For $h \in \{1, 2, 3\}$ construct the set $F_h = F(P_h)$ as above. Then there is a T_r-colouring of G from its lists iff $F_1 \cup F_2 \cup F_3 \neq L(u) \times L(w)$.

So suppose there is no T_r-colouring of G from its lists, i.e.

$$F_1 \cup F_2 \cup F_3 = L(u) \times L(w). \tag{3.10}$$

By examining the sets F_1, F_2 and F_3 we will eventually deduce that $L(u) = L(w)$, contradicting the hypothesis of the lemma.

Henceforth we will write $R = 2r+2$, and $L(u) = \{x_1, x_2, \ldots, x_R\}$ and $L(w) = \{y_1, y_2, \ldots, y_R\}$, with the elements arranged in ascending order. The following pair of properties of the sets F_h is crucial to the proof of the lemma:

$$i \leqslant j \text{ and } (x_1, y_i) \in F_h \Rightarrow (x_R, y_j) \notin F_h, \tag{3.11}$$

$$i \leqslant j \text{ and } (x_i, y_1) \in F_h \Rightarrow (x_j, y_R) \notin F_h. \tag{3.12}$$

Proof of (3.11) *and* (3.12): For $h = 1, 2, 3$ let $m = a, b, c$, respectively.

Suppose $(x_1, y_i) \in F_h$ and $(x_R, y_j) \in F_h$. So by (3.1), $I_{2m}(x_1) \neq \emptyset \neq I_{2m}(x_R)$ and by (3.6), for $1 \leqslant p \leqslant 2m-1$,

$$|L(v_p) \cap I_p(x_1)| \geqslant |I_{p+1}(x_1)| \geqslant |I_{2m}(x_1)| > 0,$$

and similarly, $L(v_p) \cap I_p(x_R) \neq \emptyset$.

Now $x_R - x_1 \geqslant 2r+1$ since $|L(u)| = 2r+2$, and so

$$\max\ I_1(x_1) = x_1 + r < x_R - r = \min\ I_1(x_R). \tag{3.13}$$

Since $L(v_1) \cap I_1(x_1) \neq \emptyset \neq L(v_1) \cap I_1(x_R)$,

$$\min L(v_1) \leqslant \max\ I_1(x_1) < \min\ I_1(x_R) \leqslant \max L(v_1),$$

and so $\min L(v_1) \notin I_1(x_R)$ and $\max L(v_1) \notin I_1(x_1)$. Again, $\max L(v_1) - \min L(v_1) \geqslant 2r+1$ since $|L(v_1)| = 2r+2$, and so

$$\max\ I_2(x_R) = \min\ L(v_1) + r < \max\ L(v_1) - r = \min\ I_2(x_1).$$

Since $L(v_2) \cap I_2(x_1) \neq \emptyset \neq L(v_2) \cap I_2(x_R)$,

$$\min\ L(v_2) \leqslant \max\ I_2(x_R) < \min\ I_2(x_1) \leqslant \max\ L(v_2),$$

and so $\min\ L(v_2) \notin I_2(x_1)$ and $\max\ L(v_2) \notin I_2(x_R)$. Repeating the above process, we see that for $1 \leqslant q \leqslant m$,

$$\max\ I_{2q-1}(x_1) < \min\ I_{2q-1}(x_R)$$

and

$$\max\ I_{2q}(x_R) < \min\ I_{2q}(x_1).$$

Since $y_i \in I_{2m}(x_1)$ and $y_j \in I_{2m}(x_R)$, it follows that $y_i > y_j$ and thus $i > j$, which establishes (3.11). The proof of (3.12) is very similar. □

It follows from (3.11) and (3.12) that no two of (x_1, y_1), (x_1, y_R) and (x_R, y_R) can belong to the same F_h, and similarly if (x_1, y_R) is replaced by (x_R, y_1). So we may assume that

$$(x_1, y_1) \in F_1, \quad (x_R, y_R) \in F_2, \quad (x_1, y_R) \in F_3 \quad \text{and} \quad (x_R, y_1) \in F_3.$$

We need a final definition: the *border* of $L(u) \times L(w)$ is defined as

$$B = \{(x, y) \in L(u) \times L(w) : x \in \{x_1, x_R\} \text{ or } y \in \{y_1, y_R\}\},$$

it consists of the starred cells in the following array:

	x_1	x_2	$\cdots$	x_R
y_1	*	*	*	*
y_2	*			*
$\vdots$	*			*
y_R	*	*	*	*

It follows from (3.11), (3.12) and convexity ((3.4) and (3.5)) that

$$F_1 \cap B = \{(x_s, y_1) : 1 \leqslant s \leqslant i\} \cup \{(x_1, y_t) : 1 \leqslant t \leqslant j\} \tag{3.14}$$

for some values $1 \leqslant i, j \leqslant 2r+1$. Now we use Lemma 3.2 to bound the size of the set $F_1 \cap B$. It can be seen that if $i + j > 2r + 2$, (3.14) precisely contradicts the statement of Lemma 3.2 for the path P_1. Hence $i + j \leqslant 2r + 2$, and $|F_1 \cap B| \leqslant 2r + 1$.

By symmetry (interchanging the roles of u and w, I_p and J_p, etc.) we can deduce the corresponding properties of F_2:

$$\begin{aligned} F_2 \cap B = {} & \{(x_s, y_R) : R+1-k \leqslant s \leqslant R\} \\ & \cup \{(x_R, y_t) : R+1-l \leqslant t \leqslant R\}. \end{aligned} \tag{3.15}$$

Since Lemma 3.2 is equally valid with the word ‘maximum’ substituted for ‘minimum’, we deduce that $k + l \leqslant 2r + 2$, and hence, $|F_2 \cap B| \leqslant 2r + 1$.

Finally, we consider $F_3 \cap B$. Since we are assuming (3.10) holds, F_3 must contain those elements of B not in F_1 or F_2, that is,

$$\begin{aligned} F_3 \cap B \supseteq \{(x_s, y_1) &: i+1 \leqslant s \leqslant R\} \\ \cup \{(x_R, y_t) &: 1 \leqslant t \leqslant R-l\} \\ \cup \{(x_1, y_t) &: j+1 \leqslant t \leqslant R\} \\ \cup \{(x_s, y_R) &: 1 \leqslant s \leqslant R-k\}. \end{aligned} \tag{3.16}$$

Note that (3.11) and (3.12), respectively, imply the following:

$$(x_1, y_t) \in F_3 \Rightarrow (x_R, y_t) \notin F_3$$

and

$$(x_s, y_1) \in F_3 \Rightarrow (x_s, y_R) \notin F_3,$$

which together with (3.16) show that $i+1 > R-k$ and $j+1 > R-l$, so that $i+k \geqslant 2r+2$ and $j+l \geqslant 2r+2$. But since $i+j \leqslant 2r+2$ and $k+l \leqslant 2r+2$, we must in fact have equality throughout (as well as in (3.16)):

$$i+j = k+l = i+k = j+l = 2r+2.$$

The third and second terms in (3.16), respectively, give $\{y_{j+1}, \ldots, y_R\} \subseteq I_{2c}(x_1)$ and $\{y_1, \ldots, y_{R-l}\} \subseteq I_{2c}(x_R)$ (recalling that the path P_3 has length $2c$), so that $|I_{2c}(x_1)| \geqslant R-j = i$ and $|I_{2c}(x_R)| \geqslant R-l = j$. Repeated application of (3.6) then shows that, for $p \in [1, 2c-1]$,

$$|L(v_p) \cap I_p(x_1)| \geqslant i \quad \text{and} \quad |L(v_p) \cap I_p(x_R)| \geqslant j. \tag{3.17}$$

However, since $I_1(x_1) \cap I_1(x_R) = \emptyset$, the sum of the left-hand sides in (3.17) when $p=1$ is at most $|L(v_1)| = R = i+j$, and so equality must hold (and $L(v_1) \subseteq I_1(x_1) \cup I_1(x_R)$). In particular, by (3.13), this tells us that $\min\, L(v_1) \in I_1(x_1)$ and $\max\, L(v_1) \in I_1(x_R)$. We now use the 'exchange' trick as in the proof of Lemma 3.2, using the facts that $(x_1, y_R) \in F_3$ and $(x_i, y_R) \in F_3$, by (3.16):

$$\begin{aligned} \min L(v_1) \in I_1(x_1) &\Rightarrow \max L(v_1) \notin I_1(x_1) \\ &\Rightarrow \max L(v_1) \in J_1(y_R) \quad \text{by (3.3)} \\ &\Rightarrow \min L(v_1) \notin J_1(y_R) \\ &\Rightarrow \min L(v_1) \in I_1(x_i) \quad \text{by (3.3)} \\ &\Rightarrow \min L(v_1) \geqslant x_i - r \qquad (3.18) \\ &\Rightarrow |L(v_1) \cap I_1(x_1)| \leqslant (2r+1) - (x_i - x_1) \\ &\leqslant 2r+2-i = j. \qquad (3.19) \end{aligned}$$

Combined with (3.17) this gives $i \leqslant j$. But similar reasoning using the facts that $\max L(v_1) \in I_1(x_R)$, $(x_R, y_1) \in F_3$ and $(x_{i+1}, y_1) \in F_3$ shows that also $j \leqslant |L(v_1) \cap I_1(x_R)| \leqslant i$. Hence we must have equality throughout:

$$i = j = k = l = r+1. \tag{3.20}$$

Fig. 2 summarises what we know about each $F_h \cap B$ at this point.

The conclusion (3.20) shows that we must also have equality in (3.18) and (3.19), as well as in (3.17) for each p. Hence $\min L(v_1) = x_{r+1} - r = x_1$ and $|L(v_1) \cap I_1(x_1)| = r+1$, and symmetrically, $\max L(v_1) = x_{r+2} + r = x_R$ and $|L(v_1) \cap I_1(x_R)| = r+1$. Thus we deduce that

$$L(u) = L(v_1) = [x_1, x_1 + r] \cup [x_R - r, x_R].$$

	x_1	$\cdots$	x_{r+1}	x_{r+2}	$\cdots$	x_R
y_1	F_1	F_1	F_1	F_3	F_3	F_3
$\vdots$	F_1					F_3
y_{r+1}	F_1					F_3
y_{r+2}	F_3					F_2
$\vdots$	F_3					F_2
y_R	F_3	F_3	F_3	F_2	F_2	F_2

Fig. 2. The intersection of the border B of $L(u) \times L(w)$ with each F_h.

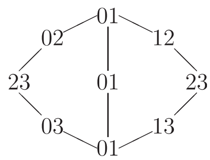

Fig. 3. A list assignment showing that $\mathrm{ch}(\Theta_{2,4,4}) > 2$.

We can now use the definitions of $I_p(x)$ directly:

$$I_2(x_1) = \bigcap_{c \in [x_R - r, x_R]} [c - r, c + r] = [x_R - r, x_R]$$

and

$$I_2(x_R) = \bigcap_{c \in [x_1, x_1 + r]} [c - r, c + r] = [x_1, x_1 + r].$$

Since we have equality in (3.17), $|L(v_2) \cap I_2(x_1)| = |L(v_2) \cap I_2(x_R)| = r + 1$, and it follows that $L(v_2) = L(v_1)$ as above. Continuing this process we see that $L(u) = L(v_1) = L(v_2) = \cdots = L(v_{2c-1}) = L(w)$, which completes the proof of Lemma 3.3. $\square$

It follows from the above proof that if there is no T_r-colouring of $G = \Theta_{2a,2b,2c}$ from lists of size $2r + 2$, the list assignment must in fact be constant on one of the paths P_1, P_2 or P_3. Note that we can indeed have T_r-ch$(G) > 2r + 2$ if $b > 1$ and $c > 1$. A list assignment illustrating this for $r = 0$ and $G = \Theta_{2,4,4}$ is shown in Fig. 3, which generalises to every $r \in \mathbb{N}$ if each colour i in each list is replaced with the interval $[i(r + 1), i(r + 1) + r]$.

Proof of Theorem 1.5. First, we observe that the technique of pruning vertices of degree 1 also works for T_r-choosability. Suppose G is assigned lists of size $2r + 2$, and we remove a vertex v of degree 1 and colour $G - v$ from its lists. We must choose a colour for v not contained in the interval $[c(w) - r, c(w) + r]$, where w is the sole neighbour of v in G, which we can do because the size of this interval is $2r + 1$. Thus if H is the core of G,

$$T_r\text{-ch}(G) \leqslant 2r + 2 \iff T_r\text{-ch}(H) \leqslant 2r + 2.$$

Applying Theorem 3.1, it suffices to consider the cases $H = K_1$, $H = C_{2m+2}$ $(m \geqslant 1)$, and $H = \Theta_{2,2,2m}$ $(m \geqslant 1)$. If $H = K_1$ then G is 1-degenerate (i.e. a tree), and the result follows from Theorem 1.3. We now establish the claims T_r-ch$(C_{2m+2}) \leqslant 2r + 2$ and T_r-ch$(\Theta_{2,2,2m}) \leqslant 2r + 2$ simultaneously by induction on m.

Note that $C_{2m+2} \subset \Theta_{2,2,2m}$, and so at each step we need only show that T_r-ch$(H) \leqslant 2r + 2$ for $H = \Theta_{2,2,2m}$. Let the vertices of H be assigned lists $L(v)$ of size $2r + 2$. Then by Lemma 3.3, we know H has a T_r-colouring if $L(u) \neq L(w)$. So assume $L(u) = L(w)$, and form the graph H' from H by identifying the vertices u and w, to give a new vertex u'

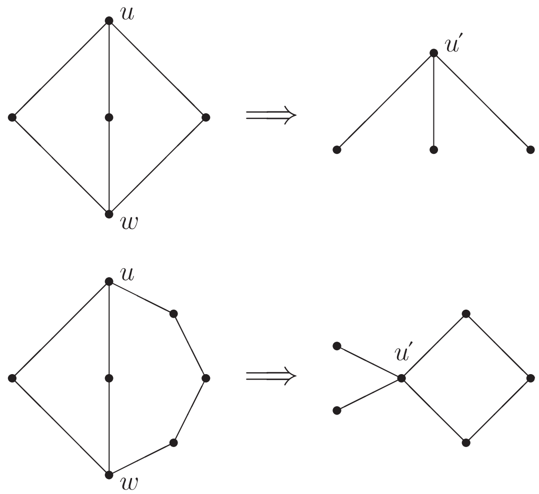

Fig. 4. Forming H' from $H = \Theta_{2,2,2m}$ (for $m = 1, 2$).

with $N(u') = N(u) \cup N(w)$, and set $L(u') = L(u)$. Then any T_r-colouring of H' will yield a T_r-colouring of H by setting $c(u) = c(w) = c(u')$.

If $m = 1$, then $H' \cong K_{1,3}$ (see Fig. 4, top) and H' has a T_r-colouring by Theorem 1.3. If $m > 1$, then $\mathrm{core}(H') \cong C_{2m}$ (Fig. 4, bottom), and H' has a T_r-colouring by the inductive hypothesis. $\square$

Remark. The case $T_r\text{-ch}(C_{2m+2}) \leqslant 2r + 2$ was first established by Tesman ([9], Theorem 3.10), and we can avoid the inductive argument in the above proof by using this result.

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